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Three-dimensional approximation of the total force on uncharged spheres in electric fields

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Abstract

Droplets on outdoor high-voltage equipment suffer a total force which is non-vanishing in general. We consider a model problem of an uncharged and conductive sphere. We show that the total force can be given as a series of inhomogeneity indicators of the undisturbed electric field in the absence of the droplet. The proof of the series involves several aspects of the spherical harmonics in the three-dimensional Fourier technique. The series expansion establishes a relation between the solutions of two Poisson's equations on different domains. It is found that the expansion converges fast. The results are applied for droplets on a realistically shaped insulator.

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1. Introduction

1.1. Applicatory and physical background

Here, we investigate the total force **F** acting on a three-dimensional undeformable uncharged and conductive sphere in an electric field. The force **F** is given as a series expansion in terms of the undisturbed electric field in the absence of the sphere. The terms vanish if the undisturbed electric field is homogeneous and thus, they are called inhomogeneity indicators.

This problem is initiated by the investigation of an outdoor high-voltage equipment which is exposed to rain and moisture. Rainwater droplets on insulators lead to undesired currents or even flashovers, which affect the insulating and hydrophobic properties of the material [1, 2]. The prevention of undesired currents and flashovers is aggravated by moving droplets leaving water films on the ageing insulator material [3].

A rainwater droplet is deformed by a force density on its surface which depends nonlinearly on the gradient of the electric field. Models and simulations can be found in [4–6]. Rainwater droplets are electrically uncharged and conductive [1, 7]. Experiments have

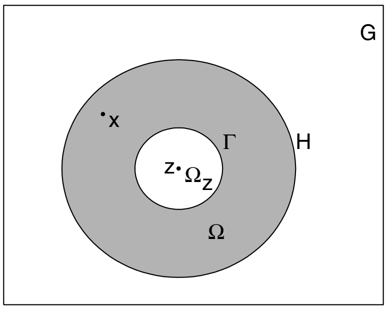

Figure 1. The sphere occupies $\Omega_\mathbf{z}$ with the boundary Γ. It is surrounded by the domain Ω with the outer boundary H inside a larger domain G.

shown that the resulting total force suffices to move a droplet [8]. This motion has been roughly modelled in [9] by a sequence of steady-state computations of the electric field and the resulting total forces.

The computation of the force acting on the droplet requires the numerical solution of Poisson's equation for the electric field in a large domain around the relatively small droplet [10, 11]. A deformable droplet model leads to a free boundary-value problem on the free capillary surface of the droplet [12, 13] which is yet more expensive to handle numerically [7]. Furthermore, the simulation of a moving droplet demands repeated calculations or the simulation of a full time-dependent model. Similar problems occur in the investigations of ferromagnetic fluids [14, 15].

A single solution of Poisson's equation for the undisturbed electric field yields the inhomogeneity indicators at all positions. The determination of $\mathbf{F}$ reduces to the evaluation of the series expansion. Its first term gives an approximation of $\mathbf{F}$ which is sufficient for the most of the practical applications. Of course, the error of the neglection of the deformability of the droplets is to be estimated in the practice.

After the introduction of the basic notations and relations, the Fourier expansion in spherical harmonics is discussed in section 2. Auxiliary facts about orthogonal functions are proven in appendix. Generalizations, convergence results and first examples are presented in section 3. The paper concludes with the presentation of further results for droplets on realistically shaped insulators and with concluding remarks in section 4.

1.2. Basic notations and relations

We denote the points of the three-dimensional Euclidean space $\mathbb{R}^3$ by $\mathbf{x} = (x_1, x_2, x_3)^\mathrm{T}$. Let us regard the ball $\Theta \subset \mathbb{R}^3$ with $\Theta = \{\mathbf{x} : \|\mathbf{x}\| < r_\mathrm{H}\}$ and the Euclidean norm $\|\cdot\|$. The boundary of the ball is $\partial\Theta = \mathrm{H}$. The sphere occupies the ball $\Omega_\mathbf{0} = \{\mathbf{x} : \|\mathbf{x}\| < r_\Gamma\}$ with $r_\Gamma < r_\mathrm{H}$ and $\partial\Omega_\mathbf{0} = \Gamma$. We introduce the ring domain $\Omega = \Theta\backslash\bar{\Omega}_\mathbf{0}$ with $\partial\Omega = \Gamma \cup \mathrm{H}$. The particular choice of the position of the sphere is generalized in section 3.

The electric field disturbed by the sphere has the potential $u(\mathbf{x})$, $\mathbf{x} \in \Omega$, and the undisturbed electric field has the potential $v(\mathbf{x})$, $\mathbf{x} \in \Theta$. The relative dielectricity is normalized in our investigations. The ball Θ has no net charge. The electric fields both yield the boundary conditions $u(\mathbf{x}) = v(\mathbf{x}) = g(\mathbf{x})$ at the boundary $\mathbf{x} \in \mathrm{H}$, figure 1. We get the boundary-value problem:

$$\Delta u(\mathbf{x}) = 0 \qquad \text{for} \quad \mathbf{x} \in \Omega, \tag{1}$$

$$u(\mathbf{x}) = g(\mathbf{x}) \qquad \text{for} \quad \mathbf{x} \in \mathrm{H}, \tag{2}$$

$$u(\mathbf{x}) = c \qquad \text{for} \quad \mathbf{x} \in \Gamma, \tag{3}$$

$$\int_\Gamma \nabla u(\mathbf{x}) \mathbf{n} \, \mathrm{d}a = 0. \tag{4}$$

Equation (1) contains Gauss's law that the divergence of the dielectric displacement equals the vanishing charge density; equation (2) is the Dirichlet boundary data. Equation (3) expresses the conductivity of the sphere which leads to a constant potential inside it. Finally, equation (4) with the normal $\mathbf{n} = \mathbf{x}/\|\mathbf{x}\|$ on Γ describes the fact that the sphere itself is free of the electric charge [10, 4]. This condition determines the constant c in equation (3). The total force is the integral of the force density over Γ and thus

$$\mathbf{F} = \frac{\varepsilon_0}{2} \int_\Gamma |\nabla u(\mathbf{x})|^2 \mathbf{n} \, \mathrm{d}a, \tag{5}$$

with the dielectricity constant ε_0. On the other hand, the undisturbed electric potential v is given by the Dirichlet boundary-value problem for Poisson's equation

$$\Delta v(\mathbf{x}) = 0 \qquad \text{for} \quad \mathbf{x} \in \Theta, \tag{6}$$

$$v(\mathbf{x}) = g(\mathbf{x}) \qquad \text{for} \quad \mathbf{x} \in \mathrm{H}. \tag{7}$$

In the following, we will show that $\mathbf{F}$ can be given as a series of inhomogeneity indicators of the potential v of the undisturbed electric field which are defined in the following.

Definition 1.1. *The inhomogeneity indicators* $\mathbf{I}_\ell^{(v)}(\mathbf{x})$ $(\ell = 1, 2, \ldots)$ *of a sufficiently smooth function* $v : \mathbb{R}^3 \to \mathbb{R}$ *at* $\mathbf{x} \in \mathbb{R}^3$ *are*

$$\mathbf{I}_\ell^{(v)}(\mathbf{x}) = \nabla(\nabla^\ell v(\mathbf{x}) : \nabla^\ell v(\mathbf{x})) \in \mathbb{R}^3, \tag{8}$$

where : *denotes the full tensor contraction.*

We remind that the full tensor contraction of two tensors of rank ℓ which are $A = \left(a_{j_1\ldots j_\ell}\right) \in \mathbb{R}^{3^\ell}$ and $B = \left(b_{j_1\ldots j_\ell}\right)$ in Cartesian coordinates is defined by $A : B = \sum_{j_1=1}^3 \cdots \sum_{j_\ell=1}^3 a_{j_1\ldots j_\ell} b_{j_1\ldots j_\ell}$.

We will prove in theorem 2.1 that

$$\mathbf{F} = \varepsilon_0 \pi \sum_{\ell=1}^\infty \frac{2^{\ell+1} r_\Gamma^{2\ell+1}}{(2\ell)!} \frac{\mathbf{I}_\ell^{(v)}(\mathbf{0})}{(1 - b^{2\ell+1})(1 - b^{2\ell+3})}, \tag{9}$$

with $b = r_\Gamma / r_\mathrm{H}$ holds. Equation (9) allows us to express the total force $\mathbf{F}$ in local terms of the undisturbed potential v at the centre $\mathbf{0}$ of the sphere. Subsection 3.1 discusses the role of the quotient b and establishes a relation which is independent of b.

We present several auxiliary lemmas in the appendix, which are needed to establish the series expansion (9), respectively, equation (10) in section 2. We use spherical coordinates (r, ϑ, φ) with $x_1 = r \sin\vartheta \cos\varphi$, $x_2 = r \sin\vartheta \sin\varphi$ and $x_3 = r \cos\vartheta$. It holds $\|\mathbf{x}\| = r$.

2. Series expansion of the total force

Theorem 2.1. *With* $b = r_\Gamma / r_\mathrm{H}$ *holds*

$$\mathbf{F} = \varepsilon_0 \pi \sum_{\ell=1}^\infty \frac{2^{\ell+1} r_\Gamma^{2\ell+1}}{(2\ell)!} \frac{\mathbf{I}_\ell^{(v)}(\mathbf{0})}{(1 - b^{2\ell+1})(1 - b^{2\ell+3})}. \tag{10}$$

Proof. Points on the boundary H have the form $\mathbf{x} = \mathbf{x}(r_{\mathrm{H}}, \vartheta, \varphi)$, and the function $g(\mathbf{x}) = g(\vartheta, \varphi)$ is smooth $g \in C^{\infty}(\mathrm{H})$ because it is a restriction of the potential function Φ. Hence, it has the convergent Fourier expansion:

$$g(\vartheta, \varphi) = a_{00} + \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell}^{m}(\vartheta, \varphi) \qquad \text{with} \qquad a_{\ell m} = \int_{\partial B} g(\mathbf{x}) \bar{Y}_{\ell}^{m}(\mathbf{x})\, \mathrm{d}a.$$

Since $g(\vartheta, \varphi) \in \mathbb{R}$, it holds $a_{\ell m} = \bar{a}_{\ell, -m}$ for all admissible ℓ and m. A function u fulfilling equation (1) has the form [18]

$$u(r, \vartheta, \varphi) = a_{00} + \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} \left(a_{\ell m}^{+} r^{\ell} + a_{\ell m}^{-} r^{-(\ell+1)} \right) Y_{\ell}^{m}(\vartheta, \varphi). \tag{11}$$

Equation (2) in the form $u(r_{\mathrm{H}}, \vartheta, \varphi) = g(\vartheta, \varphi)$ and equation (3) in the form $u(r_{\Gamma}, \vartheta, \varphi) = c$ lead to

$$a_{\ell m}^{+} r_{\mathrm{H}}^{\ell} + a_{\ell m}^{-} r_{\mathrm{H}}^{-(\ell+1)} = a_{\ell m} \qquad \text{and} \qquad a_{\ell m}^{+} r_{\Gamma}^{\ell} + a_{\ell m}^{-} r_{\Gamma}^{-(\ell+1)} = 0 \qquad \text{for} \qquad \ell \neq 0$$

after a comparison of the coefficients. The coefficient for $\ell = 0$ and $m = 0$ yields $a_{00} = c$. A simple calculation gives

$$a_{\ell m}^{+} = \frac{r_{\Gamma}^{-(\ell+1)} a_{\ell m}}{r_{\mathrm{H}}^{\ell} r_{\Gamma}^{-(\ell+1)} - r_{\mathrm{H}}^{-(\ell+1)} r_{\Gamma}^{\ell}} \qquad \text{and} \qquad a_{\ell m}^{-} = \frac{-r_{\Gamma}^{\ell} a_{\ell m}}{r_{\mathrm{H}}^{\ell} r_{\Gamma}^{-(\ell+1)} - r_{\mathrm{H}}^{-(\ell+1)} r_{\Gamma}^{\ell}}.$$

With the abbreviations $b = r_{\Gamma}/r_{\mathrm{H}}$ and

$$A_{\ell m} = \frac{(2\ell+1) r_{\Gamma}^{-2}}{r_{\mathrm{H}}^{\ell} r_{\Gamma}^{-(\ell+1)} - r_{\mathrm{H}}^{-(\ell+1)} r_{\Gamma}^{\ell}} a_{\ell m} = \frac{2\ell+1}{r_{\mathrm{H}}^{\ell}} \cdot \frac{r_{\Gamma}^{\ell-1}}{1 - b^{2\ell+1}} a_{\ell m}, \tag{12}$$

we find at the boundary Γ:

$$\frac{\partial}{\partial \mathbf{n}} u(r_{\Gamma}, \vartheta, \varphi) = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} A_{\ell m} Y_{\ell}^{m}(\vartheta, \varphi).$$

This sum does not contain any constant terms, and equation (4) is fulfilled automatically. Equation (5) reads now as

$$\mathbf{F} = \frac{\varepsilon_0}{2} \int_0^{2\pi} \int_0^{\pi} \left(\sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} A_{\ell m} Y_{\ell}^{m} \right) \left(\sum_{\ell'=1}^{\infty} \sum_{m'=-\ell'}^{\ell'} \bar{A}_{\ell' m'} \bar{Y}_{\ell'}^{m'} \right) \mathbf{n} r_{\Gamma}^{2} \sin\vartheta \, \mathrm{d}\vartheta \, \mathrm{d}\varphi.$$

Exchanging integration and summation leads to

$$\mathbf{F} = \frac{\varepsilon_0}{2} \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} \sum_{\ell'=1}^{\infty} \sum_{m'=-\ell'}^{\ell'} A_{\ell m} \bar{A}_{\ell' m'} r_{\Gamma}^{2} \int_0^{2\pi} \int_0^{\pi} Y_{\ell}^{m} \bar{Y}_{\ell'}^{m'} \mathbf{n} \sin\vartheta \, \mathrm{d}\vartheta \, \mathrm{d}\varphi. \tag{13}$$

Due to lemma A.4, the integral term in equation (13) vanishes for all ℓ and ℓ' with $|\ell - \ell'| \neq 1$. We divide the sum in equation (13) into one with $\ell' = \ell + 1$ and into another one with $\ell = \ell' + 1$. In the last one, we exchange the role of ℓ and ℓ', and we change the summation order by replacing m by $-m$ and m' by $-m'$. Then, we get

$$\mathbf{F} = \varepsilon_0 \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} \sum_{m'=-\ell'}^{\ell'} A_{\ell m} A_{\ell+1, m'} r_{\Gamma}^{2} \int_{\partial B} Y_{\ell}^{m} \bar{Y}_{\ell+1}^{m'} \mathbf{n} \, \mathrm{d}a.$$

Using theorem A.5 and equation (12), we find

$$\mathbf{F} = \varepsilon_0 \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} \sum_{m'=-\ell'}^{\ell'} \frac{b^{2\ell+1}}{b_{\ell}} a_{\ell m} \bar{a}_{\ell+1, m'} \frac{2^{\ell+2}\pi}{(2\ell)!} \nabla^{\ell} Z_{\ell}^{m} : \nabla^{\ell+1} \bar{Z}_{\ell+1}^{m'}.$$

with $b_\ell = (1 - b^{2\ell+1})(1 - b^{2\ell+3})$. This equation can be written as

$$\mathbf{F} = \varepsilon_0 \sum_{\ell=1}^{\infty} \frac{2^{\ell+2}\pi}{(2\ell)!} \frac{r_\Gamma^{2\ell+1}}{b_\ell} \nabla^\ell \sum_{m=-\ell}^{\ell} \frac{a_{\ell m}}{r_\mathrm{H}^\ell} Z_\ell^m(\mathbf{0}) : \nabla^{\ell+1} \sum_{m'=-\ell}^{\ell} \frac{a_{\ell+1,m'}}{r_\mathrm{H}^{\ell+1}} Z_{\ell+1}^{m'}(\mathbf{0}). \qquad (14)$$

Otherwise, we remark analogously to equation (11) that the solution v of the problem (6), (7) is

$$v(\mathbf{x}) = a_{00} + \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} \frac{a_{\ell m}}{r_\mathrm{H}^\ell} Z_\ell^m(\mathbf{x}),$$

and the term $\sum_{m=-\ell}^{\ell} a_{\ell m} r_\mathrm{H}^{-\ell} Z_\ell^m$ is the only summand of v, the ℓth derivative of which is not vanishing in the origin. Thus, equation (14) is identical to

$$\mathbf{F} = \varepsilon_0 \sum_{\ell=1}^{\infty} \frac{2^{\ell+2}\pi}{(2\ell)!} \frac{r_\Gamma^{2\ell+1}}{b_\ell} \nabla^\ell v(\mathbf{0}) : \nabla^{\ell+1} v(\mathbf{0}).$$

Since $\nabla^\ell v(\mathbf{0}) : \nabla^{\ell+1} v(\mathbf{0}) = \frac{1}{2}\mathbf{I}_\ell^{(v)}(\mathbf{0})$, the last equation is the searched relation (10). □

3. Convergence investigations and generalizations

3.1. Generalization

Let us regard an undisturbed electric potential Ψ defined in a certain, not necessarily round domain. The electric potential is determined by a charge distribution ρ and possible boundary conditions.

We put the sphere of radius r_Γ into the domain. Its centre is the position $\mathbf{z}$. Since the sphere is free of electric charge, it holds $r_\Gamma < d(\mathbf{z}, \operatorname{supp}\rho) = \min_{\mathbf{x}\in\operatorname{supp}\rho} \|\mathbf{z} - \mathbf{x}\|$, where $\|\cdot\|$ again denotes the Euclidean norm. The disturbed potential is named Φ. The total force acting on the sphere is $\mathbf{F}(\mathbf{z})$.

The investigations in section 2 are invariant to a translation by $\mathbf{z}$ and thus, equation (9) which has been proven in theorem 2.1, can be applied for the respective domains $\Omega' = \Omega + \mathbf{z}$ and $\Theta' = \Theta + \mathbf{z}$ if $r_\mathrm{H} < d(\mathbf{z}, \operatorname{supp}\rho)$, too.

Now, we find that the disturbed potential u in Ω' is a restriction of Φ, i.e. $u = \Phi|_{\Omega'}$. The undisturbed potential v fulfils the same boundary conditions like u at the translated boundary $\mathrm{H}' = \mathrm{H} + \mathbf{z}$, i.e. $v|_{\mathrm{H}'} = u|_{\mathrm{H}'} = \Phi|_{\mathrm{H}'}$. But that means $v|_{\mathrm{H}'} \neq \Psi|_{\mathrm{H}'}$ and $v \neq \Psi|_{\Theta'}$ in general.

The analogue to equation (9) is

$$\mathbf{F}(\mathbf{z}) = \varepsilon_0 \pi \sum_{\ell=1}^{\infty} \frac{2^{\ell+1} r_\Gamma^{2\ell+1}}{(2\ell)!} \frac{\mathbf{I}_\ell^{(v)}(\mathbf{z})}{(1 - b^{2\ell+1})(1 - b^{2\ell+3})}, \qquad (15)$$

with $b = r_\Gamma / r_\mathrm{H}$. It is remarkable that equation (15) is a local relation between the total force $\mathbf{F}$ acting on the sphere with centre $\mathbf{z}$ and the inhomogeneity indicators of v at the position $\mathbf{z}$.

But the quotient b entering equation (15) has not any physical relevance because the domain Θ' is an auxiliary domain without any physical relevance too. On the other hand, the disturbance of the electric potential by the sphere fades out with the reciprocal of the distance from the sphere, i.e. $|\Phi(\mathbf{x}) - \Psi(\mathbf{x})| \sim 1/\|\mathbf{z} - \mathbf{x}\|$ for an increasing distance $\|\mathbf{z} - \mathbf{x}\|$ from the centre $\mathbf{z}$. Hence $\Phi|_{\mathrm{H}'} - \Psi|_{\mathrm{H}'} \sim 1/r_\mathrm{H}$ in any C^n norm [10, 19], and v approaches Ψ while b tends to zero. Thus, the relation

$$\mathbf{F}(\mathbf{z}) = \varepsilon_0 \pi \sum_{\ell=1}^{\infty} \frac{2^{\ell+1} r_\Gamma^{2\ell+1}}{(2\ell)!} \mathbf{I}_\ell^{(\Psi)}(\mathbf{z}) \qquad (16)$$

is established. Here, the force $\mathbf{F}(\mathbf{z})$ is expressed by inhomogeneity indicators $\mathbf{I}_\ell^{(\Psi)}(\mathbf{z})$ of the original undisturbed potential Ψ in the absence of the sphere.

3.2. Examples

Here, we consider two examples. First, we investigate the inhomogeneity indicators for an electric potential of a point charge Q at the origin $\mathbf{x} = \mathbf{0}$, and second, we consider a line charge q along the x_3-axis.

Example 1. The potential induced by a point charge Q in the origin is $\Psi(\mathbf{z}) = \varepsilon_0^{-1} Q\gamma(\mathbf{z})$ with the fundamental solution $\gamma(\mathbf{z}) = 1/(4\pi\|\mathbf{z}\|)$ of the Laplacian in $\mathbb{R}^3$. We find the inhomogeneity indicators

$$\mathbf{I}_\ell^{(\Psi)}(\mathbf{z}) = -\frac{Q^2}{\varepsilon_0^2} \cdot \frac{(\ell+1)(2\ell)!}{2^{\ell+3}\pi^2} \cdot \frac{\mathbf{z}}{\|\mathbf{z}\|^{2\ell+4}}. \tag{17}$$

Using equation (17), we find by equation (16) the total force

$$\mathbf{F}(\mathbf{z}) = -\mathbf{z}\frac{Q^2}{4\varepsilon_0\pi} \sum_{\ell=1}^{\infty} \frac{\ell+1}{\|\mathbf{z}\|^3} \left(\frac{r_\Gamma}{\|\mathbf{z}\|}\right)^{2\ell+1} = -\mathbf{z}\frac{Q^2}{4\varepsilon_0\pi} \cdot \frac{r_\Gamma^3\left(2\|\mathbf{z}\|^2 - r_\Gamma^2\right)}{\|\mathbf{z}\|^4\left(\|\mathbf{z}\|^2 - r_\Gamma^2\right)^2}. \tag{18}$$

We remark that the total force acts towards the point charge in the origin and is independent of its sign. It decreases with $1/\|\mathbf{z}\|^5$ for increasing $\|\mathbf{z}\|$, and it decreases with r_Γ^3 for decreasing r_Γ. Furthermore, we see that the summands in the series in equation (18) decrease geometrically with the factor $(r_\Gamma/\|\mathbf{z}\|)^2 + \epsilon$ with an arbitrary $\epsilon > 0$. The ratio test for the series in powers of $(r_\Gamma/\|\mathbf{z}\|)^2$ yields convergence for all $r_\Gamma < \|\mathbf{z}\|$.

Example 2. Similarly, the potential induced by a line charge q along the x_3-axis is

$$\Psi(\mathbf{z}) = \frac{q}{2\pi\varepsilon_0} \ln\sqrt{z_1^2 + z_2^2} \qquad \text{with} \qquad \mathbf{I}_\ell^{(\Psi)}(\mathbf{z}) = -\frac{q^2}{\varepsilon_0^2} \cdot \frac{\ell(\ell-1)!}{2^{2-\ell}\pi^2} \cdot \frac{\mathbf{z}}{R^{2\ell+2}},$$

$\mathbf{z} = (z_1, z_2, z_3)^{\mathrm{T}}$ and $R = \sqrt{z_1^2 + z_2^2}$. We name $\mathbf{z}_{1,2} = \mathbf{z} - (\mathbf{z}^{\mathrm{T}}\mathbf{e}_3)\mathbf{e}_3 = (z_1, z_2, 0)^{\mathrm{T}}$ the projection of $\mathbf{z}$ onto the (z_1, z_2) plane. It holds $\|\mathbf{z}_{1,2}\| = R$. With that and equation (16), we get

$$\mathbf{F}(\mathbf{z}) = -\frac{\mathbf{z}_{1,2}}{R} \cdot \frac{q^2}{\varepsilon_0\pi} \sum_{\ell=1}^{\infty} \frac{2^{2\ell-1}\ell(\ell-1)!^2}{(2\ell)!} \left(\frac{r_\Gamma}{R}\right)^{2\ell+1} = -\mathbf{z}_{1,2}\frac{q^2}{\varepsilon_0\pi} \frac{r_\Gamma^2 \arcsin(r_\Gamma/R)}{R^2\sqrt{R^2 - r_\Gamma^2}}. \tag{19}$$

Again, $\mathbf{F}(\mathbf{z})$ is independent of the sign of the line charge, and the total force acts towards the line $z_3 = 0$. The summands in the series in equation (19) decay with the factor $(r_\Gamma/R)^2$ in essence. Again, the ratio test for the series in powers of $(r_\Gamma/R)^2$ yields convergence for all $r_\Gamma < R$. The total force itself decreases with $1/R^3$ for increasing R and with r_Γ^3 for a decreasing radius of the sphere.

3.3. Convergence behaviour

Here, we show that the decay behaviour of the inhomogeneity indicators, observed in the examples of the previous subsection, is a general property for all potentials Ψ and thus of the series in equation (16).

Theorem 3.1. *The charge distribution $\rho \in L_1(\mathbb{R}^3)$ generates the electric field with the potential Ψ. Let $R = \min_{\mathbf{x}\in\operatorname{supp}\rho}\|\mathbf{z} - \mathbf{x}\| > 0$. Then holds*

$$\left\| \mathbf{I}_\ell^{(\Psi)}(\mathbf{z}) \right\| \leqslant \|\rho\|^2_{L_1(\mathbb{R}^3)} \cdot \frac{(2\ell)!\sqrt{(\ell+1)(2\ell+1)}}{2^{\ell+3}\pi^2} \cdot \frac{\sqrt{3}}{R^{2\ell+3}} \tag{20}$$

Proof. By Green's formula holds

$$\Psi(\mathbf{z}) = \int_{\mathbb{R}^3} \rho(\mathbf{x})\gamma(\mathbf{z}-\mathbf{x})\,\mathrm{d}^3x,$$

and thus

$$\mathbf{I}_\ell^{(\Psi)}(\mathbf{z}) = \nabla_\mathbf{z} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \rho(\mathbf{x})\rho(\mathbf{y})\nabla_\mathbf{z}^\ell\gamma(\mathbf{z}-\mathbf{x}) : \nabla_\mathbf{z}^\ell\gamma(\mathbf{z}-\mathbf{y})\,\mathrm{d}^3x\,\mathrm{d}^3y. \tag{21}$$

Since we know from equation (17) that

$$\mathbf{I}_\ell^{(\gamma)}(\mathbf{z}) = \frac{(2\ell)!}{2^{\ell+4}\pi^2}\nabla\frac{1}{\|\mathbf{z}\|^{2\ell+2}},$$

we can very roughly estimate the absolute value of every ℓth derivative of the fundamental solution γ. For $\ell_1+\ell_2+\ell_3=\ell$, we find

$$\left|\frac{\partial^\ell\gamma(\mathbf{z})}{\partial z_1^{\ell_1}\partial z_2^{\ell_2}\partial z_3^{\ell_3}}\right| \leqslant \frac{1}{\pi\,\|\mathbf{z}\|^{\ell+1}}\sqrt{\frac{(2\ell)!}{2^{\ell+4}}}. \tag{22}$$

We introduce this estimation into equation (21); we get the factor 2 by differentiation using the product rule and the factor $\sqrt{3}$ by estimating the Euclidean norm by the maximum norm, and we find

$$\left\| \mathbf{I}_\ell^{(\Psi)}(\mathbf{z}) \right\| \leqslant 2\sqrt{3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|\rho(\mathbf{x})||\rho(\mathbf{y})|\frac{(2\ell)!}{2^{\ell+4}}\cdot\frac{\sqrt{(\ell+1)(2\ell+1)}}{\pi^2R^{2\ell+3}}\,\mathrm{d}^3x\,\mathrm{d}^3y \tag{23}$$

because $R \leqslant \|\mathbf{z}-\mathbf{x}\|$ and $R \leqslant \|\mathbf{z}-\mathbf{y}\|$ for all non-vanishing integrands. Equation (23) contains the assertion. □

The use of equation (20) in the series expansion (16) yields

$$\|\mathbf{F}(\mathbf{z})\| \leqslant \frac{\varepsilon_0\sqrt{3}}{4\pi}\cdot\frac{\|\rho\|^2_{L_1(\mathbb{R}^3)}}{R^2}\sum_{\ell=1}^{\infty}\left(\frac{r_\Gamma}{R}\right)^{2\ell+1}\sqrt{(\ell+1)(2\ell+1)}. \tag{24}$$

We remark, that the series in equation (24) is finite for all $r_\Gamma < R$ and converges as faster as smaller is r_Γ relative to R. Due to the roughness of the estimation (22), the upper bound (24) is a very rough estimation for $\|\mathbf{F}(\mathbf{z})\|$ too.

3.4. Observations

In the most of the practical applications, the radius r_Γ of the sphere or a general test body is decisively smaller than the distance R to the next electric charge. For instance, a droplet on an outdoor high-voltage insulator realistically has a diameter less than 1 cm. At the same time, its distance to the wire or to the support is in the range of metres. Therefore, only the first term of the series expansion (16) shall be used in realistic applications. We get

$$\mathbf{F}(\mathbf{z}) \approx 2\varepsilon_0\pi r_\Gamma^3\mathbf{I}_1^{(\Psi)}(\mathbf{z}).$$

Let us regard $\mathbf{z}=\mathbf{0}$, and let us use the mean value of $\mathbf{I}_1^{(\Psi)}(\mathbf{z})$ for $\mathbf{z}\in\Omega_\mathbf{0}$. Then we find

$$\mathbf{F} \approx 2\varepsilon_0\pi r_\Gamma^3\nabla|\nabla\Psi(\mathbf{0})|^2 \approx \frac{2\varepsilon_0\pi r_\Gamma^3}{|\Omega_\mathbf{0}|}\int_{\Omega_\mathbf{0}}\nabla|\nabla\Psi(\mathbf{x})|^2\mathrm{d}^3x = \frac{3}{2}\varepsilon_0\int_\Gamma|\nabla\Psi(\mathbf{x})|^2\mathbf{n}\,\mathrm{d}a. \tag{25}$$

In equation (5), the total force $\mathbf{F}$ has been defined by an integral over the boundary Γ in terms of the disturbed potential u which fulfils $u|_\Gamma = \Phi|_\Gamma$. However, equation (25) gives an approximative relation between **F** and the related integral over Γ in terms of the undisturbed potential Ψ. The comparison with the two-dimensional result in [20] leads to the conjecture that the factor 3 in equation (25) coincides with the dimension of the problem formulation. It may be assumed that there exists an analytical relation between the total force and the shape and size of the droplet or a general test body.

The series expansion (16) has been applied to approximate the total force acting on a droplet on the surface of a realistically shaped insulator. On the one hand, a droplet lying on a support is not round and the interface between the insulator and the air is generally not free of charge. But on the other hand, environmental influences, material imperfectness and the gravity interact with the theoretically predicted total force **F**.

The first term of equation (16) shows that several attractive points are located at the thin parts of the insulator. The absolute value of the total force at the ends of the insulator is larger than in the middle part. Hence, there is a tendency that a droplet moves into the thin parts of the insulator and it prefers the ends. That coincides with the observations in [1, 3].

The approximation of **F** is compared with a stationary approach for deformable two-dimensional droplet models in [9]. The computation of the total force acting on a droplet at a fixed position requires the solution of a free boundary-value problem, i.e. of a sequence of equations of type (1, . . . ,4) [4]. There, the approximation was checked to be a rather good one by a comparison of **F** from equation (16) with the results from the computational approach by free boundary-value problems. But now, using the series (16) in the newly introduced inhomogeneity indicators, only one computation of the undisturbed potential Ψ in the neighbourhood of the insulator is necessary to approximate the total force at every position **z** in the computational domain.

4. Conclusion

We have stated a series expansion which gives the total force **F** acting on a conductive and electrically uncharged sphere in an electric field. The terms of the series are multiples of the newly introduced inhomogeneity indicators of the undisturbed electric field in the absence of the sphere. The inhomogeneity indicators $\mathbf{I}_\ell^{(\Psi)}$, $\ell \in \mathbb{N}\backslash\{0\}$, describe the local deviation of the undisturbed electric field with the potential Ψ from being homogeneous. The series are local relations between the total force and the undisturbed electric field at the centre of the sphere.

The series converges quickly. For practical applications, its first term seems to be sufficient to get a reasonable approximation for the total force on more generally shaped bodies.

The force has been given for two examples, first, of a point charge and second, of a line charge. It acts towards the electric charge independently of its sign, and it decays rather fast with the distance from the electric charge. Nevertheless, the effect suffices to explain the tendency of the droplets's motion on the surface of a realistically shaped insulator. There, the droplets are attracted by the thin parts of the insulator, and the attraction is enforced at its ends. This results coincide with practical observations.

Further work is required to established a respective relation providing a comparably good approximation of **F** for more generally shaped test bodies and for test bodies near to electric charges. Nevertheless, the proposed approximation of the total force acting on droplets may serve to optimize the shape of outdoor high-voltage equipment.

Appendix. Auxiliary facts about orthogonal functions

We name the Legendre polynomials for $\ell \in \mathbb{N}$ by

$$P_\ell(z) = \frac{1}{2^\ell \ell!} \frac{\mathrm{d}^\ell}{\mathrm{d}z^\ell}(z^2-1)^\ell.$$

These are the Jacobi polynomials for $\alpha = \beta = 0$. The associated Legendre functions for $0 \leqslant m \leqslant \ell$ are

$$P_\ell^m(z) = \sqrt{\frac{(\ell-m)!}{(\ell+m)!}} \sqrt{\frac{2\ell+1}{2}} (1-z^2)^{\frac{m}{2}} \frac{\mathrm{d}^m}{\mathrm{d}z^m} P_\ell(z).$$

The spherical harmonics are

$$Y_\ell^m(\vartheta, \varphi) = \frac{1}{\sqrt{2\pi}} P_\ell^{|m|}(\cos\vartheta)\, \mathrm{e}^{\mathrm{i}m\varphi} \tag{A.1}$$

for $-\ell \leqslant m \leqslant \ell$ [16]. It holds $Y_\ell^m = \bar{Y}_\ell^{-m}$. We name the related homogeneous polynomials

$$Z_\ell^m(\mathbf{x}) = Z_\ell^m(x_1, x_2, x_3) = r^\ell Y_\ell^m(\vartheta, \varphi)$$

for $-\ell \leqslant m \leqslant \ell$. They have the polynomial degree ℓ, i.e. $Z_\ell^m \in \mathrm{Hom}_\ell(\mathbb{R}^3)$. We name the unit ball $B := \{\mathbf{x} : r < 1\}$. We use the Kronecker symbol $\delta_{\ell,k}$ which equals 1 for $\ell = k$ and vanishes else. Furthermore, we introduce the abbreviations $d_\ell := \sqrt{(2\ell+3)/(2\ell+1)}$ and $k_{\ell,m} := \sqrt{(\ell+m)(\ell+m+1)}$.

Lemma A.1. *Let* $P_\ell^{(m)}(z) = \frac{\mathrm{d}^m}{\mathrm{d}z^m} P_\ell(z)$ *be the mth derivatives of the Legendre polynomials. For all* ℓ *and* m *holds*

$$(\ell - m + 1) P_{\ell+1}^{(m)}(z) - z P_{\ell+1}^{(m+1)}(z) + P_\ell^{(m+1)}(z) = 0, \tag{A.2}$$

and for $m \geqslant 1$ *holds*

$$m P_{\ell+1}^{(m)}(z) = \tfrac{1}{2}\left[k_{\ell,m}^2 P_\ell^{(m-1)}(z) + (1-z^2) P_\ell^{(m+1)}(z)\right]. \tag{A.3}$$

Proof. Since the derivatives $P_\ell^{(m)}(z)$ are Jacobi polynomials $P_{\ell-m}^{(\alpha,\beta)}$ for $\alpha = \beta = m$, we have the recursion formula [17]

$$(\ell+m) P_{\ell-1}^{(m)}(z) + (\ell-m+1) P_{\ell+1}^{(m)}(z) - (2\ell+1) z P_\ell^{(m)}(z) = 0 \tag{A.4}$$

and the differentiation formula [17]

$$(\ell - m + 1) z P_{\ell+1}^{(m)}(z) + (1-z^2) P_{\ell+1}^{(m+1)}(z) = (\ell+m+1) P_\ell^{(m)}(z). \tag{A.5}$$

We differentiate equation (A.5) and subtract the same equation after the replacement of m by $m+1$. Hence, we get equation (A.2).

Now, we multiply equation (A.2) by z and subtract the product form equation (A.5). We find

$$P_{\ell+1}^{(m+1)}(z) - (\ell+m+1) P_\ell^{(m)}(z) - z P_\ell^{(m+1)}(z) = 0. \tag{A.6}$$

The replacement of ℓ by $\ell - 1$ in equation (A.5) yields

$$(1-z^2) P_\ell^{(m+1)}(z) = (\ell+m) P_{\ell-1}^{(m)}(z) - (\ell-m) z P_\ell^{(m)}(z), \tag{A.7}$$

and the replacement of m by $m-1$ in equation (A.6) leads to

$$(\ell+m) P_\ell^{(m-1)}(z) = P_{\ell+1}^{(m)}(z) - z P_\ell^{(m)}. \tag{A.8}$$

Equation (A.8) is multiplied by the factor $\ell + m + 1$ and added to equation (A.7), and we get

$$k_{\ell,m}^2 P_\ell^{(m-1)}(z) + (1 - z^2) P_\ell^{(m+1)}(z)$$
$$= (\ell + m) P_{\ell-1}^{(m)}(z) + (\ell + m + 1) P_{\ell+1}^{(m)}(z) - (2\ell + 1) z P_\ell^{(m)} = 2m P_{\ell+1}^{(m)}(z),$$

whereas the last identity follows from the recursion formula (A.4). This is equation (A.3). □

The next lemma describes the relation of the partial derivatives of $Z_{\ell+1}^m \in \mathrm{Hom}_{\ell+1}(\mathbb{R}^3)$ to the homogeneous polynomials $Z_\ell^m \in \mathrm{Hom}_\ell(\mathbb{R}^3)$ which correspond to the spherical harmonics of decreased order. In the formulae occur Z_ℓ^m with $|m| > \ell$ albeit with the factor $k_{\ell,-(\ell+1)} = 0$. For completeness, we define $Z_\ell^m \equiv 0$ for all m with $|m| > \ell$. Furthermore, the case $m = 0$ does not follow the formulae given in equations (A.9) and (A.10). We will mention it later in equation (A.17), the proof of which follows completely analogous ideas.

Lemma A.2. *For all ℓ and m with $1 \leqslant |m| \leqslant \ell + 1$ holds*

$$\frac{\partial}{\partial x_1} Z_{\ell+1}^m = \frac{1}{2} d_\ell \left(k_{\ell,m} Z_\ell^{m-1} - k_{\ell,-m} Z_\ell^{m+1} \right), \tag{A.9}$$

$$\frac{\partial}{\partial x_2} Z_{\ell+1}^m = \frac{-\mathrm{i}}{2} d_\ell \left(k_{\ell,m} Z_\ell^{m-1} + k_{\ell,-m} Z_\ell^{m+1} \right), \tag{A.10}$$

and for all ℓ and m with $|m| \leqslant \ell + 1$ holds

$$\frac{\partial}{\partial x_3} Z_{\ell+1}^m = d_\ell \sqrt{(\ell+1)^2 - m^2} Z_\ell^m. \tag{A.11}$$

Proof. We prove the relations for non-negative m. In the case of negative m, they follow immediately by conjugation. The homogeneous polynomials Z_ℓ^m can be written as

$$Z_\ell^m(x_1, x_2, x_3) = \frac{1}{\sqrt{2\pi}} r^\ell P_\ell^{|m|}\left(\frac{x_3}{r}\right) \left(\frac{x_1 + \mathrm{i}x_2}{\sqrt{x_1^2 + x_2^2}} \right)^m. \tag{A.12}$$

We abbreviate $x = x_1/r$, $y = x_2/r$ and $z = x_3/r$ with the property $x^2 + y^2 + z^2 = 1$. We multiply equation (A.2) by x^2 and add equation (A.3), and due to $1 - x^2 = (x^2 z^2 + y^2)/(1 - z^2)$, we get

$$(\ell+1) x^2 P_{\ell+1}^{(m)} + \frac{m x^2 z^2}{1 - z^2} P_{\ell+1}^{(m)} - x^2 z P_{\ell+1}^{(m+1)} + \frac{m y^2}{1 - z^2} P_{\ell+1}^{(m)}$$
$$= \frac{1}{2} \left(k_{\ell,m}^2 P_\ell^{(m-1)} - (x^2 - y^2) P_\ell^{(m+1)} \right). \tag{A.13}$$

After the multiplication of equation (A.2) with xy, we can note equation (A.13) in the form

$$xy(\ell+1) P_{\ell+1}^{(m)} + \frac{m x y z^2}{1 - z^2} P_{\ell+1}^{(m)} - xyz P_{\ell+1}^{(m+1)} - \frac{m x y}{1 - z^2} P_{\ell+1}^{(m)} = -xy P_\ell^{(m+1)}. \tag{A.14}$$

Now, equation (A.13) is the real part and equation (A.14) is the imaginary part of the equation

$$(x + \mathrm{i}y) \left[(\ell+1) x P_{\ell+1}^{(m)} + \frac{m x z^2}{1 - z^2} P_{\ell+1}^{(m)} - xz P_{\ell+1}^{(m+1)} - \frac{\mathrm{i} m y}{1 - z^2} P_{\ell+1}^{(m)} \right]$$
$$= \frac{1}{2} \left[k_{\ell,m}^2 P_\ell^{(m-1)} - (x + \mathrm{i}y)^2 P_\ell^{(m+1)} \right].$$

We multiply this equation by $(1-z^2)^{\frac{m-1}{2}}$, we use

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[(1-z^2)^{\frac{m}{2}} P_{\ell+1}^{(m)}(z)\right] = -mz(1-z^2)^{\frac{m}{2}-1} P_{\ell+1}^{(m)} + (1-z^2)^{\frac{m}{2}} P_{\ell+1}^{(m+1)},$$

and we replace $x^2+y^2 = 1-z^2$ in the denominator. Then, we find

$$\frac{x+\mathrm{i}y}{\sqrt{x^2+y^2}}\left\{\left[(\ell+1)x - \frac{\mathrm{i}my}{x^2+y^2}\right](1-z^2)^{\frac{m}{2}} P_{\ell+1}^{(m)} - xz\frac{\mathrm{d}}{\mathrm{d}z}\left[(1-z^2)^{\frac{m}{2}} P_{\ell+1}^{(m)}\right]\right\}$$
$$= \frac{1}{2}\left(k_{\ell,m}^2(1-z^2)^{\frac{m-1}{2}} P_\ell^{(m-1)} - \frac{(x+\mathrm{i}y)^2}{x^2+y^2}(1-z^2)^{\frac{m+1}{2}} P_\ell^{(m+1)}\right).$$

We replace again $x = x_1/r$, $y = x_2/r$ and $z = x_3/r$ and multiply the last equation, first, by $\frac{1}{\sqrt{2\pi}}\sqrt{\frac{(\ell+1-m)!}{(\ell+1+m)!}}$ and, second, by $\sqrt{(2\ell+3)/2}$. The last factor is used in the form $d_\ell\sqrt{(2\ell+1)/2}$ on the right-hand side. Together with the derivative with respect to x_1 from equation (A.12) which is

$$\frac{\partial Z_{\ell+1}^m}{\partial x_1} = \frac{r^\ell(x_1+\mathrm{i}x_2)^m}{2\pi\left(x_1^2+x_2^2\right)^{\frac{m}{2}}}\left[\left((\ell+1)\frac{x_1}{r} - \frac{\mathrm{i}mx_2r}{x_1^2+x_2^2}\right)P_{\ell+1}^m - \frac{x_1x_3}{r^2}P_{\ell+1}^{m'}\right] \tag{A.15}$$

we get the searched equation (A.9). The proof of equation (A.10) follows similar steps with changed roles of x_1 and x_2, respectively, x and y. Here, we continue with a proof of equation (A.11) which is technically less expensive.

We multiply the differentiation formula (A.5) with $(1-z^2)^{\frac{m}{2}}$ and with $\sqrt{\frac{(\ell-m+1)!}{(\ell+m+1)!}}$, and we find

$$(\ell+1)zP_{\ell+1}^m(z) + (1-z^2)\frac{d}{dz}P_{\ell+1}^m(z) = d_\ell\sqrt{(\ell+1)^2-m^2}P_\ell^m(z).$$

The replacement of z by x_3/r and the multiplication by r^ℓ lead to

$$(\ell+1)x_3r^{\ell-1}P_{\ell+1}^m\left(\frac{x_3}{r}\right) + r^\ell\left(1-\frac{x_3^2}{r^2}\right)P_{\ell+1}^{m'}\left(\frac{x_3}{r}\right) = d_\ell\sqrt{(\ell+1)^2-m^2}r^\ell P_\ell^m\left(\frac{x_3}{r}\right),$$

and after the reformulation of the left-hand side to

$$\frac{\partial}{\partial x_3}\left[r^{\ell+1}P_{\ell+1}^m\left(\frac{x_3}{r}\right)\right] = d_\ell\sqrt{(\ell+1)^2-m^2}r^\ell P_\ell^m\left(\frac{x_3}{r}\right). \tag{A.16}$$

The multiplication of equation (A.16) by $\frac{1}{\sqrt{2\pi}}\mathrm{e}^{\mathrm{i}m\varphi}$ provides the searched equation (A.11).

□

In the case $m=0$, formulae similar to equations (A.9) and (A.10) are found. For instance, it holds

$$\frac{\partial}{\partial x_1}Z_{\ell+1}^0 = \frac{1}{2}d_\ell k_{\ell,0}\left(Z_\ell^{-1}+Z_\ell^1\right), \tag{A.17}$$

what can be proven by equations (A.15) and (A.3).

The following lemma expands the orthogonality of the spherical harmonics to the higher gradients of the related homogeneous polynomials. This allows us to operate easily with the full tensor contraction in equation (8).

Lemma A.3. *For all ℓ, m, and m' with $m, m' \leqslant |\ell|$ holds*

$$\nabla^\ell Z_\ell^m(\mathbf{x}) : \nabla^\ell \bar{Z}_\ell^{m'}(\mathbf{x}) = \delta_{m,m'}\frac{(2\ell+1)!}{2^{\ell+2}\pi}. \tag{A.18}$$

Proof. Since the homogeneous polynomials Z_ℓ^m are of the form $Z_\ell^m = r^\ell Y_\ell^m$ with $Y_\ell^m : \partial B \to \mathbb{R}$, there are functions $Y_{\ell,j}^m : \partial B \to \mathbb{R}^{3^j}$ with the property

$$\nabla^j Z_\ell^m = r^{\ell-j} Y_{\ell,j}^m .$$

Since $Z_\ell^m \in \mathrm{Hom}_\ell(\mathbb{R}^3)$, the functions $\nabla^j Z_\ell^m$ and $Y_{\ell,j}^m$ are constant for $j = \ell$ and vanishing for $j > \ell$.

The volume element is denoted by $\mathrm{d}^3x = r^2 \sin\vartheta\, \mathrm{d}r\, \mathrm{d}\vartheta \mathrm{d}\varphi$ and the surface element is $\mathrm{d}a = \sin\vartheta \mathrm{d}\vartheta \mathrm{d}\varphi$ on ∂B. Now, we apply partial integration on

$$R_\ell := \int_B \nabla^j Z_\ell^m : \nabla^j \bar{Z}_\ell^{m'}\, \mathrm{d}^3x = \int_{\partial B} \nabla^{j-1} Z_\ell^m : \frac{\partial}{\partial \mathbf{n}} \nabla^{j-1} \bar{Z}_\ell^{m'}\, \mathrm{d}a - \int_B \nabla^{j-1} Z_\ell^m : \nabla^{j-1} \Delta \bar{Z}_\ell^{m'}\, \mathrm{d}^3x.$$

The last term vanishes because $Z_\ell^{m'}$ is a potential function, and we get

$$R_\ell = \int_{\partial B} r^{\ell-j+1} Y_{\ell,j-1}^m : \left(\frac{\mathrm{d}}{\mathrm{d}r} r^{\ell-j+1}\right) \bar{Y}_{\ell,j-1}^{m'}\, \mathrm{d}a = (\ell - j + 1) \int_{\partial B} Y_{\ell,j-1}^m : \bar{Y}_{\ell,j-1}^{m'}\, \mathrm{d}a. \qquad \text{(A.19)}$$

Otherwise holds

$$R_\ell = \int_B r^{\ell-j} Y_{\ell,j}^m : r^{\ell-j} \bar{Y}_{\ell,j}^{m'}\, \mathrm{d}^3x = \int_0^1 r^{2(\ell-j)} r^2\, \mathrm{d}r \int_{\partial B} Y_{\ell,j}^m : \bar{Y}_{\ell,j}^{m'}\, \mathrm{d}a,$$

where the term r^2 is a part of the metric tensor for the volume integral, and we find

$$R_\ell = \frac{1}{2(\ell - j) + 3} \int_{\partial B} Y_{\ell,j}^m : \bar{Y}_{\ell,j}^{m'}\, \mathrm{d}a. \qquad \text{(A.20)}$$

The identity of the right-hand sides of equations (A.19) and (A.20) is applied for $j = 1, \ldots, \ell$. Since $\nabla^\ell Z_\ell^m$ is constant, it holds

$$4\pi \nabla^\ell Z_\ell^m : \nabla^\ell \bar{Z}_\ell^{m'} = \int_{\partial B} Y_{\ell,\ell}^m : \bar{Y}_{\ell,\ell}^{m'} da = \ell! \cdot 3 \cdot 5 \cdot \cdots \cdot (2\ell + 1) \int_{\partial B} Y_\ell^m : \bar{Y}_\ell^{m'}\, \mathrm{d}a.$$

The orthogonality of the spherical harmonics yields equation (A.18). □

The next lemma gives the properties of the spherical harmonics which are needed in the proof of theorem A.5. Again, the case $m = 0$ has to be handled separately as it has already been done in lemma A.2.

Lemma A.4. *For all $\ell' \geqslant \ell$, m with $1 \leqslant |m| \leqslant \ell$ and m' with $|m'| \leqslant \ell'$ holds*

$$\int_{\partial B} Y_\ell^m \bar{Y}_{\ell'}^{m'} \sin\vartheta \cos\varphi\, \mathrm{d}a = \frac{\delta_{\ell',\ell+1} d_\ell}{2(2\ell+3)} (\delta_{m',m+1} k_{\ell+1,m} - \delta_{m'+1,m} k_{\ell+1,-m}), \qquad \text{(A.21)}$$

and for all $\ell' \geqslant \ell$, m with $|m| \leqslant \ell$ and m' with $|m'| \leqslant \ell'$ holds

$$\int_{\partial B} Y_\ell^m \bar{Y}_{\ell'}^{m'} \cos\vartheta\, \mathrm{d}a = \delta_{m,m'} \delta_{\ell',\ell+1} \frac{\sqrt{(\ell+1)^2 - m^2}}{\sqrt{(2\ell+1)(2\ell+3)}}. \qquad \text{(A.22)}$$

Proof. We use definition (A.1) of the spherical harmonics, and we find that the left-hand side of equation (A.21) is decomposed into two integrals over ϑ and φ. After the substitution $z = \cos\vartheta$, we get

$$\int_{\partial B} Y_\ell^m \bar{Y}_{\ell'}^{m'} \sin\vartheta \cos\varphi\, \mathrm{d}a = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{\mathrm{i}(m-m')\varphi} \cos\varphi\, \mathrm{d}\varphi \int_{-1}^1 P_\ell^{|m|}(z) P_{\ell'}^{|m'|}(z) \sqrt{1 - z^2}\, \mathrm{d}z.$$

The first factor on the right-hand side is non-vanishing only in the case $|m - m'| = 1$. We set $m' = m + 1$, and otherwise, the order of the factors is exchanged. Now, we investigate the second factor. We restrict on $m \geqslant 0$ without loss of generality, and we get

$$R = \int_{-1}^{1} P_\ell^m(z) P_{\ell'}^{m+1}(z) \sqrt{1-z^2}\,\mathrm{d}z = F \int_{-1}^{1} P_\ell^{(m)}(z) P_{\ell'}^{(m+1)}(z)(1-z^2)^{m+1}\,\mathrm{d}z, \tag{A.23}$$

with the factor

$$F = \sqrt{\frac{(\ell-m)!}{(\ell+m)!}} \sqrt{\frac{(\ell'-m-1)!}{(\ell'+m+1)!}} \frac{\sqrt{(2\ell+1)(2\ell'+1)}}{2}.$$

Since $(1-z^2)^{m+1}$ in equation (A.23) has zeros of multiplicity $m+1$ at $z = \pm 1$, multiple partial integration of R yields

$$R = (-1)^{m+1} F \int_{-1}^{1} P_{\ell'}(z) \frac{\mathrm{d}^{m+1}}{\mathrm{d}z^{m+1}} \left[(1-z^2)^{m+1} P_\ell^{(m)}(z)\right] \mathrm{d}z \tag{A.24}$$

and

$$R = (-1)^{m} F \int_{-1}^{1} P_{\ell}(z) \frac{\mathrm{d}^{m}}{\mathrm{d}z^{m}} \left[(1-z^2)^{m+1} P_{\ell'}^{(m+1)}\right] \mathrm{d}z. \tag{A.25}$$

Since the $(m+1)$th derivative in equation (A.24) is a polynomial of degree $\ell+1$, i.e. an element of $\Pi_{\ell+1}$ and the mth derivative in equation (A.25) is an element of $\Pi_{\ell'+1}$, R is non-vanishing only for $\ell - 1 \leqslant \ell' \leqslant \ell + 1$ due to the orthogonality of the Legendre polynomials to all polynomials of lower degree. It is vanishing in the case $\ell = \ell'$, too, because then, the integrand in equation (A.23) is a product of an odd and an even function which is $(1-z^2)^{m+1}$.

Regarding the case $\ell' = \ell + 1$, we investigate the coefficient before $z^{\ell+1}$ in the derivative in equation (A.25), and we get

$$\frac{\mathrm{d}^{m+1}}{\mathrm{d}z^{m+1}} \left[(1-z^2)^{m+1} P_\ell^{(m)}(z)\right] = \frac{(-1)^{m+1}}{2\ell+1} \frac{(\ell+m+2)!}{(\ell-m)!} P_{\ell+1}(z) + r_\ell(z),$$

where $r_\ell \in \Pi_\ell$. Now, we find

$$\int_{-1}^{1} P_\ell^m(z) P_{\ell+1}^{m+1}(z) \sqrt{1-z^2}\,\mathrm{d}z = \frac{2F(\ell+m+2)!}{(2\ell+1)(\ell-m)!(2\ell+3)} = \frac{d_\ell k_{\ell+1,m}}{2\ell+3}.$$

Analogously, the case $\ell' = \ell - 1$ is handled, and after a displacement of the exponents by -1, we get

$$\int_{-1}^{1} P_\ell^m(z) P_{\ell+1}^{m-1}(z) \sqrt{1-z^2}\,\mathrm{d}z = -\frac{d_\ell k_{\ell+1,-m}}{2\ell+3}.$$

Thus, equation (A.21) is proven.

When we consider equation (A.22), the integral over φ yields $\delta_{m,m'}$, and we have to investigate

$$R = \int_{-1}^{1} P_\ell^m(z) P_{\ell'}^m(z) z\,\mathrm{d}z = F \int_{-1}^{1} P_\ell^{(m)}(z) P_{\ell'}^{(m)}(z)(1-z^2)^m z\,\mathrm{d}z, \tag{A.26}$$

with

$$F = \sqrt{\frac{(\ell-m)!}{(\ell+m)!}} \sqrt{\frac{(\ell'-m)!}{(\ell'+m)!}} \frac{\sqrt{(2\ell+1)(2\ell'+1)}}{2}.$$

Again, the integrand has zeros of multiplicity m at $z = \pm 1$, and m partial integrations lead to

$$R = (-1)^m F \int_{-1}^{1} P_{\ell'}(z) \frac{\mathrm{d}^m}{\mathrm{d}z^m} \left[P_\ell^{(m)}(z)(1-z^2)^m z\right] \mathrm{d}z. \tag{A.27}$$

The mth derivative in equation (A.27) is a polynomial of degree $\ell+1$. If $\ell+1<\ell'$, then R is zero because $P_{\ell'}$ is orthogonal to all polynomials of lower degree. The exchange of ℓ and ℓ' leads to the identical assertion for $\ell'+1<\ell$. In the case $\ell=\ell'$, the square $P_\ell^m(z)P_{\ell'}^m(z)=P_\ell^m(z)^2$ in equation (A.26) is an even function, and z itself is an odd function. Thus, the integral vanishes. It remains the case $\ell'=\ell+1$. Regarding the highest coefficient, we find

$$\frac{\mathrm{d}^m}{\mathrm{d}z^m}\left[P_\ell^{(m)}(z)(1-z^2)^m z\right]=\frac{(-1)^m}{2\ell+1}\frac{(\ell+m+1)!}{(\ell-m)!}P_{\ell+1}(z)+r_\ell(z),$$

where $r_\ell\in\Pi_\ell$. Thus, we find the searched equation (A.22) by

$$\int_{-1}^{1}P_\ell^m(z)P_{\ell+1}^m z\,\mathrm{d}z=\frac{F}{2\ell+1}\frac{(\ell+m+1)!}{(\ell-m)!}\frac{2}{2\ell+3}=d_\ell\frac{\sqrt{(\ell+1)^2-m^2}}{2\ell+3}.$$

This finishes the proof. □

We remark that the replacement of the term $\cos\varphi$ by $\sin\varphi$ in equation (A.21) leads to an analogous formula where we get + instead of − and −i instead of 1 on the right-hand side.

Furthermore, the case $m=0$ is handled by

$$\int_{\partial B}Y_\ell^0(\vartheta,\varphi)\bar{Y}_{\ell'}^{m'}(\vartheta,\varphi)\sin\vartheta\cos\varphi\,\mathrm{d}a=\frac{1}{2}\frac{\delta_{\ell',\ell+1}d_\ell}{2\ell+3}\delta_{|m'|,1}k_{\ell+1,1}$$

analogously to equation (A.21), compare equation (A.17).

Theorem A.5. *For all $\ell\geqslant 0$, m and m' with $|m|\leqslant\ell$ and $|m'|\leqslant\ell+1$ holds*

$$(2\ell+1)(2\ell+3)\int_{\partial B}Y_\ell^m(\vartheta,\varphi)\bar{Y}_{\ell+1}^{m'}(\vartheta,\varphi)\mathbf{n}\,\mathrm{d}a=\frac{2^{\ell+2}\pi}{(2\ell)!}\left(\nabla^\ell Z_\ell^m(\mathbf{0}):\nabla^{\ell+1}\bar{Z}_{\ell+1}^{m'}(\mathbf{0})\right).\tag{A.28}$$

Proof. The normal is $\mathbf{n}=(\sin\vartheta\cos\varphi,\sin\vartheta\sin\varphi,\cos\vartheta)^{\mathrm{T}}$. When we regard the first component of equation (A.28), we find

$$(2\ell+3)\int_{\partial B}Y_\ell^m\bar{Y}_{\ell+1}^{m'}\sin\vartheta\cos\varphi\,\mathrm{d}a=\frac{d_\ell}{2}(\delta_{m',m+1}k_{\ell+1,m}-\delta_{m'+1,m}k_{\ell+1,-m})\tag{A.29}$$

for the left-hand side for $|m|\geqslant 1$ by lemma A.4, equation (A.21). Otherwise, the right-hand side of equation (A.28) can be transformed into

$$\frac{2^{\ell+2}\pi}{(2\ell)!}\nabla^\ell Z_\ell^m:\frac{\partial}{\partial x_1}\nabla^\ell\bar{Z}_{\ell+1}^{m'}=\frac{2^{\ell+1}\pi}{(2\ell)!}d_\ell\nabla^\ell Z_\ell^m:\nabla^\ell\left(k_{\ell,m'}\bar{Z}_\ell^{m'-1}-k_{\ell,-m'}\bar{Z}_\ell^{m'+1}\right)$$

by lemma A.2 (equation (A.9)). Furthermore, lemma A.3 yields

$$\frac{2^{\ell+2}\pi}{(2\ell)!}\nabla^\ell Z_\ell^m:\frac{\partial}{\partial x_1}\nabla^\ell\bar{Z}_{\ell+1}^{m'}=\frac{d_\ell(2\ell+1)}{2}(k_{\ell,m'}\delta_{m',m+1}-k_{\ell,-m'}\delta_{m'+1,m}).\tag{A.30}$$

After expressing m' by the only occurring $m'=m+1$ and $m'=m-1$ leading to a non-vanishing right-hand side, equation (A.30) equals equation (A.29). The case $m=0$ is handled by equation (A.17) and an analogous formula to equation (A.21). The second components of equation (A.28) follow a very similar explanation.

Lemma A.4 (equation (A.22)) gives a formulation for the third component of the left-hand side of equation (A.28) which is

$$(2\ell+1)(2\ell+3)\int_{\partial B}Y_\ell^m\bar{Y}_{\ell+1}^{m'}\cos\vartheta\,\mathrm{d}a=\delta_{m,m'}\sqrt{(\ell+1)^2-m^2}\sqrt{(2\ell+1)(2\ell+3)}.\tag{A.31}$$

The application of lemma A.2 (equation (A.11)) onto the right-hand side of equation (A.28) yields

$$\frac{2^{\ell+2}\pi}{(2\ell)!}\nabla^\ell Z_\ell^m:\frac{\partial}{\partial x_3}\nabla^\ell\bar{Z}_{\ell+1}^{m'}=\frac{2^{\ell+2}\pi}{(2\ell)!}d_\ell\sqrt{(\ell+1)^2-(m')^2}\nabla^\ell Z_\ell^m:\nabla^\ell Z_\ell^{m'},$$

which transforms by lemma A.3 into

$$\frac{2^{\ell+2}\pi}{(2\ell)!}\nabla^{\ell} Z_{\ell}^{m} : \frac{\partial}{\partial x_3}\nabla^{\ell} \bar{Z}_{\ell+1}^{m'} = d_{\ell}\delta_{m,m'}\sqrt{(\ell+1)^2 - m^2}(2\ell+1). \tag{A.32}$$

The comparison of equation (A.31) and (A.32) leads to the searched identity. □

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